



n-th order derivative

- Sequence of n-fold integrals and n-fold derivatives:

$$\dots, \int_a^t d\tau_2 \int_a^{\tau_2} f(\tau_1) d\tau_1, \int_a^t f(\tau_1) d\tau_1, f(t), \frac{df(t)}{dt}, \frac{d^2 f(t)}{dt^2}, \dots$$

- Fourier transform: $FT\left(\frac{d^n f(t)}{dt^n}\right) = (i\omega)^n F(\omega)$

$$\dots, (i\omega)^{-2} F(\omega), (i\omega)^{-1} F(\omega), F(\omega), i\omega F(\omega), (i\omega)^2 F(\omega), \dots$$

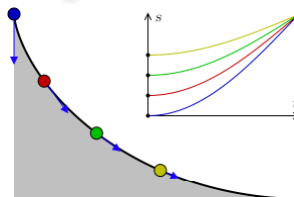
- I. Podlubny, Fractional Differential Equations, Academic Press, 1999

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First physical problem

- Niels Henrik Abel (1802 – 1829): a generalized version of the tautochrone problem (Abel's mechanical problem)
 - Requires derivative of order 0.5
- Tautochrone/isochrone curve: the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of the starting point.
- The curve is a cycloid, and the time is equal to π times the square root of the radius over the acceleration of gravity.
 - Abel, Auflösung einer mechanischen Aufgabe, J. Reine u. Angew. Math, 1826,
 - B. Holmboe, "Abel: Oeuvres complètes", 1839, IV Résolution d'un problème mécanique.



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Derivative of arbitrary order

- Derivative of order α :

$$\frac{d^\alpha f(t)}{dt^\alpha} = {}_a D_t^\alpha f(t)$$

- $\alpha < 0 \Leftrightarrow$ integration
- a and t : limits in defining integral

- Fourier transform (neglecting initial cond's):

$$FT \left(\frac{d^\alpha f(t)}{dt^\alpha} \right) = (i\omega)^\alpha F(\omega)$$

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Hand-waving approach to fractional operator (1)

- Integer ($m > \alpha$) + fraction ($\alpha - m < 0$):

$$FT \left(\frac{d^\alpha f(t)}{dt^\alpha} \right) = (i\omega)^\alpha F(\omega) = (i\omega)^m F(\omega) (i\omega)^{\alpha-m}$$

- First part: ordinary derivative
- Second part: fractional part
 - What is its Fourier transform?



Generalization: Fourier transform of $t^{-\beta}$

Fourier transform:

$$h(t) = \frac{t_0^\beta}{t^\beta}, t > 0 \Leftrightarrow H(\omega) = \Gamma(1 - \beta) (i\omega t_0)^{\beta-1}$$

- t_0 is a constant
- $\Gamma(.)$ is gamma-function
 - Factorial: $\Gamma(n+1) = n!$



Hand-waving approach to fractional operator (2)

- Fractional Fourier transform as a convolution of derivative of order first integer m larger than α and a memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} \propto \frac{d^m f(t)}{dt^m} * \frac{1}{t^{1+\alpha-m}}$$



Fractional derivative: Two definitions

- Riemann-Liouville: order $\alpha \in \mathbb{R}$, $m-1 \leq \alpha < m$:

$${}_a D_t^\alpha f(t) = \left(\frac{d}{dt}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

– First convolution, then integer order derivation

- Caputo: order $m-1 \leq \alpha < m$:

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

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First integer order derivative, then convolution

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Riemann-Liouville vs Caputo

- Riemann-Liouville requires initialization of derivatives of non-integer orders:

$$\lim_{t \rightarrow a} {}_a D_t^{\alpha-1} f(t), \lim_{t \rightarrow a} {}_a D_t^{\alpha-2} f(t), \dots$$

- Caputo requires initialization of integer order derivatives: $f^{(k)}(0)$, $k=0, 1, \dots, m-1$
 - Usually have physical meaning
 - Simpler to use in numerical solutions



Fractional derivative for numerics

- Caputo (lower limit $a = -\infty$):

$${}_C D_t^\alpha f(t) = \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial^m f(t)}{\partial t^m} * g_{m-\alpha}(t)$$

$$g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, t > 0$$

- Convolution with a memory function

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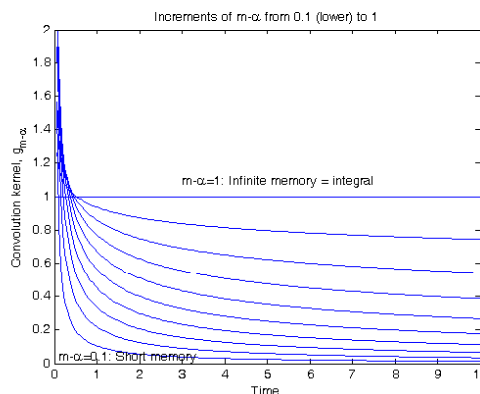


Memory function

- Convolution kernel:

$$g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} t^{\alpha+1-m}$$

- $m-\alpha = \varepsilon^+$: no memory,
 $\Gamma(\varepsilon^+) \rightarrow \infty$ for $\varepsilon^+ \rightarrow 0$
 $\Rightarrow$ kernel $\rightarrow$ impulse
- $m-\alpha = 1$: infinite memory



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Fractional derivative of order 0..1

- Example: $0 \leq \alpha < 1$ (Caputo with $m=1$):

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial f(t)}{\partial t} * g_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{f^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau$$

- Limits:

$$- \alpha \rightarrow 0 \Rightarrow \frac{\partial^\alpha f(t)}{\partial t^\alpha} \rightarrow \int_{-\infty}^t \frac{f^{(1)}(\tau)}{(t-\tau)^0} d\tau = f(t)$$

$$- \alpha \rightarrow 1 \Rightarrow \frac{\partial^\alpha f(t)}{\partial t^\alpha} \rightarrow \int_{-\infty}^t f^{(1)}(\tau) \delta(t-\tau) d\tau = f^{(1)}(t)$$



Conclusion

Two interpretations of fractional derivative:

1. Fourier:

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega)$$

2. Convolution of ordinary derivative of order $m > \alpha$ and memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} \propto \frac{d^m f(t)}{dt^m} * \frac{1}{t^{1+\alpha-m}}$$

